System of Generalized Vector Quasi-Variational-Like Inequalities with Set-Valued Mappings

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Abstract: In this paper, we introduce and study the system of generalized vector quasi-variational-like inequalities in Hausdorff topological vector spaces. By means of fixed point theorem, we obtain existence theorem for the system of generalized vector quasi-variational-like inequalities and also derive the existence result of solutions for the generalized vector quasi-variational-like inequalities.

Key word: The system of generalized vector quasi-variational-like inequalities; Fixed point theorem; Open lower section; Upper semicontinuous; Weak C-diagonal quasiconvexity.

I. INTRODUCTION

Throughout this paper, unless otherwise specified, assume that $I$ is an index set. For each $i \in I$, let $Z_i$ be a locally convex topological vector space (l.c.s., in short) and $K_i$ be a nonempty convex subset of Hausdorff topological vector space (t.v.s., in short) $E_i$. Let $Y_i$ be a subset of continuous function space $L_E Z_i$ from $E_i$ into $Z_i$, where $L_E Z_i$ is equipped with a $\sigma$-topology. Let int $A$ and co $A$ denote the interior and convex hull of a set $A$, respectively. Let $C_i : K \rightarrow Z_i$ be a set-valued mapping such that int $C_i(x) \neq \emptyset$ for each $x \in K$. Denote that $K = \prod_{i \in I} K_i$ and $E = \prod_{i \in I} E_i$.

For each $i \in I$, let $\eta_j : K_i \times K_j \rightarrow E_i$ be a vector-valued mapping, $G_i : L(E, Z) \rightarrow L(E_i, Z_i)$, $T_i : K \rightarrow Y_i$ and $D_i : K \rightarrow K_i$ be three set-valued mappings. Then, we introduce a system of generalized vector quasi-variational-like inequalities (SGVQVLI, in short) which is to find $(x, t) \in K \times Y$ such that $\forall x \in K$, $\forall t \in Y$, and $\forall y \in D_i(x)$

$\langle G_i \tilde{t}_i, \eta_j(y_i, \tilde{x}_i) \rangle \in -\text{int } C_i(\tilde{x})$,

where $\langle l, x \rangle$ denotes the evaluation of $l \in L(E, Z)$ at $x \in E$. By the corollary of Schaefer [14], $L(E, Z)$ becomes a l.c.s.. By Ding [6], the bilinear map $\langle ., . \rangle : L(K, Z) \times K \rightarrow Z$ is continuous. Present work is motivated and inspired by Peng [12, 13] on the existence of solutions for vector quasi-variational-like inequalities in Hausdorff topological vector spaces.

The above system of generalized vector quasi-variational-like inequalities encompasses many models of system of variational inequalities. For example, if $G_i$ is an identity mapping, the following problems are the special cases of SGVQVLI:

1. SGVQVLI reduces to the problem of finding $\tilde{x} \in K$ such that for each $i \in I$, $\tilde{x}_i \in D_i(\tilde{x})$ and

$\forall y_i \in D_i(\tilde{x})$, $\exists \tilde{t}_i \in T_i(\tilde{x}) : \langle \tilde{t}_i, \eta_i(y_i, \tilde{x}_i) \rangle \notin -\text{int } C_i(\tilde{x})$,

which is introduced and studied by Peng [12].

2. For each $i \in I$, $\eta_i(y_i, \tilde{x}_i) = y_i - \tilde{x}_i$, and $C_i(x) = C$, where $C_i$ is a convex, closed, pointed cone in $Z_i$ with int $C \neq \emptyset$, then the SGVQVLI reduces to the problem of finding $\tilde{x} \in K$ such that for each $i \in I$, $\tilde{x}_i \in D_i(\tilde{x})$ and

$\forall y_i \in D_i(\tilde{x})$, $\exists \tilde{t}_i \in T_i(\tilde{x}) : \langle \tilde{t}_i, y_i - \tilde{x}_i \rangle \notin -\text{int } C$. 


which is introduced and studied by Allevi et al. [3].

(3) If $I$ is a singleton set, SGVQVLI reduces to the problem of finding $\bar{x} \in K$ such that $\bar{x} \in D(\bar{x})$ and

$$\forall y \in D(\bar{x}), \exists \bar{t} \in T(\bar{x}): \langle \bar{t}, \eta(y, \bar{t}) \rangle \notin \text{int } C(\bar{x}),$$

which is introduced and studied by Ding [5].

In addition, if $D(\bar{x}) = K$ SGVQVLI reduces to the problem of finding $\bar{x} \in K$ and

$$\forall y \in K, \exists \bar{t} \in T(\bar{x}): \langle \bar{t}, \eta(y, \bar{t}) \rangle \notin \text{int } C(\bar{x}),$$

which is studied by Ahmad [2].

If $\eta(y, \bar{x}) = y - \bar{x}$ then SGVQVLI reduces to the problem of finding $\bar{x} \in K$ and

$$\forall y \in K, \exists \bar{t} \in T(\bar{x}): \langle \bar{t}, y - \bar{x} \rangle \notin \text{int } C(\bar{x}),$$

which is studied by Fang [7].

In this Paper, we introduce the system of generalized vector quasi-variational-like inequalities and by using the fixed point theorem, we establish an existence result of its solutions. As a special case, we also derive the existence result of generalized vector quasi-variational-like inequalities. Our results extend and improve some main results of Allevi et al. [3], Peng et al. [12, 13], Husain et al. [9, 10]. The technical instrument in our proof is similar to that employed by Hou et al. [8], Tian [16].

II. PRELIMINARIES

In this section, in order to prove the main result, we need the following definitions and lemmas.

**Definition 2.1** [8] Let $E$ and $Z$ be two t.v.s. and $K$ be a convex subset of t.v.s. $E$. Let $\theta: K \times K \to Z$ be two set-valued mappings. Assume given any finite subset $\Lambda = \{x_1, x_2, \ldots, x_n\}$ in $K$, any $x = \sum_{i=1}^{n} \alpha_i x_i$ with $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$. Then $\theta$ is said to be weak C-diagonally quasiconvex (WC-DQC, in short) in the second argument if for some $x_i \in \Lambda$,

$$\theta(x_i, x) \notin \text{int } C(x).$$

**Definition 2.2.** [17] Let $A$ and $B$ be two topological spaces and $T: A \to B$ be a set-valued mapping. Then,

(1) $T$ is said to have open lower sections if the set $T^{-1}(y) = \{x \in A : y \in T(x)\}$ is open in $A$ for every $y \in B$.

(2) $T$ said to be upper semicontinuous (u.s.c., in short) if for each $x \in A$ and each open set $C$ in $B$ with $T(x) \subset C$, there exists an open neighborhood $O$ of $x$ in $A$ such that $T(u) \subset C$ for each $u \in O$.

(3) $T$ is said to be closed if for any net $\{x_{\alpha}\}$ in $A$ such that $x_{\alpha} \to x$ and any net $\{y_{\alpha}\}$ in $B$ such that $y_{\alpha} \to y$ and $y_{\alpha} \in T(x_{\alpha})$ for any $\alpha$, we have $y \in T(x)$.

**Definition 2.3.** [4] Let $A$ and $B$ be two topological spaces. If $T: A \to B$ is u.s.c. set-valued mapping with closed values, then $T$ is closed.

**Lemma 2.4.** [15] Let $A$ and $B$ be two topological spaces. Suppose that $T: A \to B$ is u.s.c. mapping with compact values. Suppose $\{x_{\alpha}\}$ is a net in $A$ such that $x_{\alpha} \to x$. If $y_{\alpha} \in T(x_{\alpha})$ for each $\alpha$, then there are a $y_{\alpha} \in T(x)$ and a subnet $\{y_{\beta}\}$ of $\{y_{\alpha}\}$ such that $y_{\beta} \to y_{\alpha}$.

**Lemma 2.5.** [18] Let $A$ and $B$ be two topological spaces. If $T: A \to B$ and $K: A \to B$ are set-valued mappings having open lower sections, then

(1) a set-valued mapping $J: A \to B$ defined by, for each $x \in A$, $J(x) = \text{co } T(x)$ has open lower sections;
(2) a set-valued mapping $\theta : A \rightrightarrows B$ defined by, for each $x \in A$, $\theta(x) = T(x) \cap K(x)$ has open lower sections.

For each $i \in I$, $E_i$ a Hausdorff t.v.s., Let $\{K_i\}$ be a family of nonempty compact convex subsets with each $K_i$ in $E_i$. Let $K = \prod_{i \in I} K_i$ and $E = \prod_{i \in I} E_i$. The following system of fixed-point theorem is needed in this paper.

**Lemma 2.6.** [1] For each $i \in I$, let $T_i : K \rightrightarrows K_i$ be a set-valued mapping. Assume that the following conditions hold.

(1) For each $i \in I$, $T_i$ is convex set-valued mapping;

(2) $K = \bigcup_1 \{T_i^-(x_i) : x_i \in K_i\}$.

Then there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x}) = \prod_{i \in I} T_i(\bar{x})$, that is $\bar{x}_i \in T_i(\bar{x})$, for each $i \in I$, where $\bar{x}_i$ is the projection of $\bar{x}$ onto $K_i$.

### III. Main Results

In this section, we present existence result of a solution for the SGVQVLI by using the fixed point theorem.

**Theorem 3.1.** For each $i \in I$, let $Z_i$ be a l.c.s., $K_i$ a nonempty compact convex subset of Hausdorff t.v.s. $E_i$, $Y_i$ a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a $\sigma$-topology. For each $i \in I$, assume that the following conditions are satisfied.

(1) $D_i : K \rightrightarrows K_i$ and $T_i : K \rightrightarrows Y_i$ are two nonempty convex set-valued mappings and have open lower sections;

(2) For each $t_i \in Y_i$, and $x_i \in \text{co } \Lambda_i$, the mapping $\langle G_i t_i, \eta_i (., x_i) \rangle : K \rightrightarrows Z_i$ is WC-DQC;

(3) For each $y_i \in K_i$, the set $\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i (y_i, x_i) \rangle \subseteq \text{co } \Lambda_i (x) \}$ is open.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i \big(\bar{x}_i, \bar{x}_j\big) \rangle \subseteq \text{co } \Lambda_i (\bar{x}), \forall y_i \in D_i(\bar{x}).$$

That is, SGVQVLI has a solution $(\bar{x}, \bar{t}) \in K \times Y$.

**Proof.** Define a set-valued mapping $P_i : K \times Y \rightrightarrows K_i$ by

$$P_i(x, t) = \{y_i \in K_i : \langle G_i t_i, \eta_i (., x_i) \rangle \subseteq \text{co } \Lambda_i (x)\}, \forall (x, t) \in K \times Y. \quad (3.1)$$

We first prove that $x_i \not\in \text{co } (P_i(x, t))$ for all $(x, t) \in K \times Y$. To see this, suppose, by way of contradiction, that there exist some $i \in I$ and some point $(\bar{x}, \bar{t}) \in K \times Y$ such that $\bar{x}_i \in \text{co } (P_i(\bar{x}, \bar{t}))$. Then, these exist finite points $y_{i_1}, y_{i_2}, ..., y_{i_n}$ in $K_i$ and $\alpha_j \geq 0$ with

$$\sum_{j=1}^n \alpha_j = 1$$

such that $\bar{x}_i = \sum_{i=1}^n \alpha_j y_{i_j}$ and $y_{i_j} \in \text{co } (P_i(\bar{x}, \bar{t}))$ for all $j = 1, 2, ..., n$, such that

$$\langle G_i \bar{t}_i, \eta_i \big(\bar{x}_i, \bar{x}_j\big) \rangle \subseteq \text{co } \Lambda_i (\bar{x}), \quad j = 1, 2, ..., n. \quad (3.2)$$

Equations (3.1) and (3.2) contradict the hypothesis (2). Hence, $x_i \not\in \text{co } (P_i(x, t))$.

By hypothesis (3), for each $i \in I$ and each $y_i \in K_i$, we know that $Q_i^-(y_i) = \{ (x, t) \in K \times Y : \langle G_i t_i, \eta_i (y_i, x_i) \rangle \subseteq \text{co } \Lambda_i (x) \}$ is open and so $P_i$ has open lower sections.

For each $i \in I$, consider a set-valued mapping $Q_i : K \times Y \rightrightarrows K_i$ defined by

$$Q_i(x, t) = \text{co } \big( P_i(x, t) \big) \cap D_i(x), \quad \forall (x, t) \in K \times Y. \quad (3.3)$$
Since $D_l$ has open lower sections by hypothesis (1), we may apply Lemma 2.5 to assert that the set-valued mapping $Q_l$ has also open lower sections. With the help of (3.3), we consider

$$W_l = \left\{(x,t) \in K \times Y : Q_l(x,t) \neq \emptyset\right\} \subset K \times Y. \quad (3.4)$$

There are two cases to consider.

**Case (I)** If $W_l = \emptyset$:

From (3.4), we have $\text{co} \left\{P_l(x,t)\right\} \cap D_l(x) = \emptyset, \quad \forall \ (x,t) \in K \times Y$. This implies that, $\forall \ (x,t) \in K \times Y$,

$$P_l(x,t) \cap D_l(x) = \emptyset.$$

On the other hand, by condition (1), and the fact $K_l$ is a compact convex subset of $E_l$, we can apply Lemma 2.6 to assert the existence of a fixed point $x^*_l \in D_l(x^*)$.

Since $T_l(x^*) \neq \emptyset$, picking $t^*_l \in T_l(x^*)$, we have

$$P_l(x^*, t^*) \cap D_l(x^*) = \emptyset.$$

From (3.4), we have $\forall y_l \in D_l(x^*), \ y_l \notin P_l(x^*, t^*)$. Hence, in this particular case, the assertion of the theorem holds.

**Case (II)** If $W_l \neq \emptyset$:

Define a set-valued mapping $S_l : K \times Y \rightharpoonup K_l$ by

$$S_l(x,t) = \begin{cases} Q_l(x,t), & (x,t) \in W_l \\ D_l(x), & (x,t) \in K \times Y \setminus W_l. \end{cases} \quad (3.5)$$

From (3.5), $S_l(x,t)$ is a convex set-valued mapping and for each $u \in K_l$, $S_l^{-}(u) = Q_l^{-}(u) \cup (D_l^{-}(u) \times Y)$ is open. For each $i \in I$, consider the set-valued mapping $H_l : K \times Y \rightharpoonup K \times Y$ where $H_l = \prod_{i \in I} H_l_i$, defined by

$$H_l(x,t) = (S_l(x,t), (T_l(x))). \quad (3.6)$$

By hypothesis (1) and (3.6), $H_l$ satisfies all the conditions of Lemma 2.6. Therefore, there exists $(x^*, t^*) \in K \times Y$ such that $(x^*_l, t^*_l) \in H_l(x^*, t^*)$. Suppose that $(x^*, t^*) \in W_l$, then

$$x^*_l \in \text{co}(P_l(x^*, t^*)) \cap D_l(x^*),$$

so that $x^*_l \in \text{co}(P_l(x^*, t^*))$. This is a contradiction. Hence, $(x^*, t^*) \notin W_l$. From (3.4), (3.5), and (3.6), we have

$$(x^*_l, t^*_l) \in (D_l(x^*), T_l(x^*)), \quad \text{and} \quad Q_l(x^*, t^*) = \emptyset.$$

Thus

$$x^*_l \in D_l(x^*), \ t^*_l \in T_l(x^*), \ \text{co}(P_l(x^*, t^*)) \cap D_l(x^*) = \emptyset.$$

This implies
Consequently, the assertion of the theorem holds in this case. □

**Remark 3.2.** If *I* is a singleton set, by using Theorem 3.1, we obtain the following existence result of solutions for GVQVLI.

**Corollary 3.3.** Let *Z* be a l.c.s., *K* a nonempty compact convex subset of Hausdorff t.v.s. *E*, *Y* a nonempty compact convex subset of *L*(*E*, *Z*), which is equipped with a σ-topology. Assume that the following conditions are satisfied.

1. *D : K ↠ K* and *T : K ↠ Y* are two nonempty convex set-valued mappings and have open lower sections;
2. For each *t ∈ Y* and *x ∈ coΛ*, the mapping *⟨Gt, η(x)⟩ : K ↠ Z* is WC-DQC;
3. For each *y ∈ K*, the set *(x, t) ∈ K × Y : ⟨Gt, η(y, x)⟩ ⊆ − int C(x)∗* is open.

Then there exist a point *x ∈ D(Λ)∗* and a point *t ∈ T(Λ)∗* such that

\[ ⟨GT, η(y, x)⟩ ∉ − int C(x), ∀ y ∈ D(Λ)∗ \]

That is, GVQVLI has a solution *(x, t) ∈ K × Y*.

**REFERENCES**


